Texture, Strain, and Stress
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Texture
- Rotation tensor, pole figures, ODF
- Neutron diffraction
- Examples

Deformation (strain)
- Deformation gradient tensor
- Rotation / stretch

Stress
- Stress tensor – normal and shear components
- Stress / strain relations

Examples
- Weldments, twist tubes
- In-situ deformation experiments
Crystallographic Texture

Most engineering materials are made up of a collection of crystallites (grains), each characterized by the orientation of the unit cell.

The distribution of orientations is referred to as the *texture* of the material.
Crystallographic Texture:
The extremes and the middle ground

Single Crystal

[100]

Textured Polycrystal

[100]

Powder

[100]
Representing Texture

- Express crystallite orientation with respect to suitable material axes → Rotation tensor (set of angles)
- Function to describe volume fraction of material having unit cell orientation in a given angular range → orientation density function

\[ V_f (\Delta \omega) = \int_{\Delta \omega} g(\omega) d\omega \]
Rotation Tensor

- Express crystallite orientation with respect to a standard orientation (unrotated basis).
  - Develop a well-defined operation to go from the standard orientation to an arbitrary orientation.

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- Rotation tensor \( R \)
  - Columns contain direction cosines of rotated base vectors \( \{g_i\} \) wrt unrotated basis \( \{E_i\} \)

\[
R_{ij} = \sim_i \cdot \sim_j
\]
Rotation Tensor

\[ R_{ij} = \mathbf{E}_i \cdot \mathbf{g}_j \]

\( \mathbf{E}_i \) \( \mathbf{g}_j \)

1\textsuperscript{st} index \rightarrow \text{unrotated basis} \quad \text{OLD}

2\textsuperscript{nd} index \rightarrow \text{rotated basis} \quad \text{NEW}

\[
\begin{pmatrix}
\tilde{g}_1 \\
\tilde{g}_2 \\
\tilde{g}_3
\end{pmatrix} =
\begin{pmatrix}
(g_1)_1 & (g_2)_1 & (g_3)_1 \\
(g_1)_2 & (g_2)_2 & (g_3)_2 \\
(g_1)_3 & (g_2)_3 & (g_3)_3
\end{pmatrix} \begin{pmatrix}
\mathbf{E}_1 \\
\mathbf{E}_2 \\
\mathbf{E}_3
\end{pmatrix} \quad R_{ij} = (\tilde{g}_j)_i
\]

\( R_{ij} \) = the i\textsuperscript{th} component of vector \( \tilde{g}_j \) with respect to the unrotated basis.
Euler-Rodrigues Identity

Given a unit rotation axis \( \mathbf{a} \) and a rotation angle \( \alpha \), what is the rotation tensor?

\[
\mathbf{R} = \cos \alpha \mathbf{I} + (1 - \cos \alpha) \mathbf{aa} + \sin \alpha \mathbf{A}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + \begin{bmatrix}
a_1a_1 & a_1a_2 & a_1a_3 \\
a_2a_1 & a_2a_2 & a_2a_3 \\
a_3a_1 & a_3a_2 & a_3a_3
\end{bmatrix} + \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
\]

\[
tr_{\mathbf{R}} = R_{11} + R_{22} + R_{33} = 1 - 2 \cos \alpha
\]

\[
[\hat{a}] = \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]

with \( \sqrt{a_1^2 + a_2^2 + a_3^2} = 1 \)

\[
[\hat{a} \otimes \hat{a}] = [\hat{a} \otimes \hat{a}] = [\hat{a}][\hat{a}]^T = \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} \begin{bmatrix}
a_1 & a_2 & a_3
\end{bmatrix} = \begin{bmatrix}
a_1a_1 & a_1a_2 & a_1a_3 \\
a_2a_1 & a_2a_2 & a_2a_3 \\
a_3a_1 & a_3a_2 & a_3a_3
\end{bmatrix}
\]

\( \mathbf{A} \) is the axial tensor associated with \( \hat{a} \), defined such that \( \hat{a} \times \hat{b} = \mathbf{A} \hat{b} \)

i.e. \( \{\hat{a} \times \hat{b}\} = [\mathbf{A}]\{\hat{b}\} \)
Change of Basis
(Passive Transformation)

\[ R_{ij} = \tilde{E}_i \cdot \tilde{g}_j \begin{cases} 1^{\text{st}} \text{ index} & \rightarrow \text{unrotated basis} \\ 2^{\text{nd}} \text{ index} & \rightarrow \text{rotated basis} \end{cases} \]

\[ \tilde{E}_i = \sum_{j=1}^{3} R_{ij} \tilde{g}_j \]  
Unrotated base vector \( \tilde{E}_i \) as weighted sum of rotated base vectors \( \tilde{g}_i \).

\[ \tilde{g}_i = \sum_{j=1}^{3} R_{ji} \tilde{E}_j \]  
Rotated base vector \( \tilde{g}_i \) as weighted sum of unrotated base vectors \( \tilde{E}_i \).

\[ v_i = \sum_{j=1}^{3} R_{ji} V_j \]  
Components of a vector

\[ \mathbf{v} = v_i \tilde{g}_i = V_i \tilde{E}_i \]

\[ V_i = \sum_{j=1}^{3} R_{ij} v_j \]
Rotation of a Vector
Active Transformation

\[ \mathbf{y} = \mathbf{R} \cdot \mathbf{y} \quad \Rightarrow \quad \mathbf{v}_i = \sum_{j=1}^{3} R_{ij} \mathbf{v}_j \quad \Rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} \]

\[
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \nu \cos \alpha \\ \nu \sin \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \nu(\cos \theta \cos \alpha - \sin \theta \sin \alpha) \\ \nu(\sin \theta \cos \alpha + \cos \theta \sin \alpha) \\ 0 \end{bmatrix} = \begin{bmatrix} \nu \cos(\alpha + \theta) \\ \nu \sin(\alpha + \theta) \\ 0 \end{bmatrix}
\]

\[
\langle \nu \cos(\alpha + \theta), \nu \sin(\alpha + \theta), 0 \rangle
\]

\[
\langle \nu \cos \alpha, \nu \sin \alpha, 0 \rangle
\]

\[
\|\mathbf{v}\| = \|\mathbf{\nu}\| = \nu
\]
**Sequential Rotations – Fixed Axes**

Any rotation matrix can be expressed as three sequential rotations about the stationary laboratory axes \{X, Y, Z\}.

Consider a rotation \(R_x\) followed by a rotation \(R_y\), followed by a rotation \(R_z\):

1) Rotate about X-axis
2) Rotate about Y axis
3) Rotate about Z axis

Rotating vectors:

\[ R = R_z \cdot R_y \cdot R_x \]

\[ y = z \cdot y \cdot z \cdot x \cdot y \]
Euler angles – Sequential Rotations About Follower Axes

The rotation matrix can also be expressed as three sequential rotations about axes \( \{x, y, z\} \), which follow the material as it rotates.

1) Rotate about Z-axis
   \[
   \begin{pmatrix}
   Z \rightarrow Z \\
   X \rightarrow x' \\
   Y \rightarrow y'
   \end{pmatrix}
   \]

2) Rotate about x' axis
   \[
   \begin{pmatrix}
   Z \rightarrow z' \\
   x' \rightarrow x' \\
   y' \rightarrow y''
   \end{pmatrix}
   \]

3) Rotate about z' axis
   \[
   \begin{pmatrix}
   z' \rightarrow z' \\
   x' \rightarrow x'' \\
   y'' \rightarrow y'''
   \end{pmatrix}
   \]

The matrices are multiplied \textit{in the order of operation}, instead of in the reverse order as for rotations about the stationary laboratory axes.

The rotation matrix is the same regardless of how it is calculated!
Euler angles and Euler space-
http://core.materials.ac.uk/repository/alumatter/metallurgy/anisotropy/euler_angles_space_and_odf_revised.swf?targetFrame=EulerSpace

By Gunter Gottstein, Markus Buescher, MATTER, released under CC BY-NC-ND 2.0 license

The three Euler angles $\phi_1$, $\Phi$ and $\phi_2$ can be represented by a single point on a 3D graph - the so-called "Euler Space".

Change the orientation using the three sliders and observe the point moving in Euler Space.
The Stereographic Projection

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Stereographic projection for describing grain orientation -
http://aluminium.matter.org.uk/content/html/eng/0210-0010-swf.htm

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Points 1', 2' and 3' are drawn where these connecting lines intersect the equatorial plane. These are the 100 poles and uniquely represent the orientation of the unit cell in 3D space on a 2D plane.

Click here to continue >>
Measuring Texture

By rotating a specimen, crystallites with plane-normals in different directions rotate through the scattering vector.

Rotate the specimen over a full hemisphere of orientations to obtain a complete (100) pole figure.
Measuring Pole Figures (Intensity Maps)

- **\( \chi = 0^\circ \)**
  - \( 0^\circ < \eta < 360^\circ \)

- **\( \chi = 30^\circ \)**
  - \( 0^\circ < \eta < 360^\circ \)

- **\( \chi = 90^\circ \)**
  - \( 0^\circ < \eta < 360^\circ \)

\( \chi \)-axis is into page

\( [010] \) → \( [100] \)

\( 2\theta = 2\theta_{100} \)
$\chi = 0$

$\eta = 90$

$\chi = 90$

$\eta = 0$

RD

ND

Q at $\chi = 0$

Q at $\chi = \sim 15$

Q at $\chi = \sim 45$

Q at $\chi = 90$

TD

Centre is Z

Contour separation: 0.500 x random
Stereographic Pole Figures

- Each point on a pole figure corresponds to a family of grain orientations – those with a particular crystal direction aligned with $\mathbf{Q}$. 

[Diagram of stereographic pole figures with directions I, S, Q, and crystal axes [010], [100].]
The representation of orientations in the pole figure is ambiguous if there is more than one orientation because each orientation is determined by several poles which cannot be associated to a specific orientation in a unique way.
The Orientation Density Function

- We obtain the orientation density function (ODF) by fitting spherical harmonics to the experimentally measured pole figures.

\[ dV_f = g(\alpha)d\alpha \quad V_f(\Delta \alpha) = \int_{\Delta \alpha} g(\alpha)d\alpha \]

- The ODF tells us what fraction of grains have an orientation between \( \alpha \) and \( \alpha + d\alpha \), where \( \alpha \) represents the Euler angles.
The Orientation Density Function

- A single experimental pole figure usually is not enough!
- Cubics (≥1), HCP's (≥2), THE MORE THE MERRIER

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Deep Drawing of Aluminum Cans

A disc of Al is drawn into a die to form a can.

Preferred orientations of crystallites in the disc may create anisotropic yield surfaces.

Ears develop. The highly-automated canning-line is shut down when a large-eared can gets stuck.
Rolled-Sheet Texture of Al Disc

- The initial sheet material used to form disc has a strong rolling texture.
- Earing can be related to this initial texture.
- Carry out a systematic investigation of process route changes designed to reduce texture, or produce a more favourable texture.
Microstructural Homogeneity or Not

Neutron (0002) intensity vs. circumferential position at fixed radius in Ti-alloy impeller

Forging process of original billet, resulted in 4-fold symmetry of enhanced crystallite orientations in finished component.
Deformation

Deformation is what happens when a material changes shape and orientation.

Each particle occupies a unique position in the initial and deformed configurations so we can define a one-to-one mapping function:

\[ \tilde{\mathbf{x}} = \chi(\tilde{\mathbf{X}}) \quad \text{ and } \quad d\tilde{\mathbf{x}} = \tilde{\mathbf{F}} \cdot d\tilde{\mathbf{x}} \]

\( \tilde{\mathbf{F}} \) is the best descriptor of material deformation, because it contains all the information concerning shape changes and rotations.
The Deformation Gradient Tensor

\[ \mathbf{x} = \mathbf{x}(\mathbf{X}) \Rightarrow \begin{cases} x_1 = \chi_1(\mathbf{X}) \\ x_2 = \chi_2(\mathbf{X}) \\ x_3 = \chi_3(\mathbf{X}) \end{cases} \]

\[ d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \Rightarrow \mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{X}} \]

\[ F_{ij} = \frac{\partial x_i}{\partial X_j} \]

\[ \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \]
A material can be considered to be made up of little material cubes, characterized by the orthonormal vectors \((E_1, E_2, E_3)\) that form its edges.

After deformation, each little material cube transforms into a parallelipiped, characterized by the three vectors \((g_1, g_2, g_3)\) that form its edges.

This approximation (cube \(\rightarrow\) parallelipiped) becomes exact in the limit of an infinitely refined grid.

\(g_1, g_2,\) and \(g_3\) are generally not orthogonal, nor of unit length.
Deformation – Graphical Introduction

The $i^{th}$ column of the deformation gradient tensor contains the components of vector $g_i$ with respect to the lab basis ($E_1$, $E_2$, $E_3$).

$$g_1 = (g_{1})_1 E_1 + (g_{1})_2 E_2 + (g_{1})_3 E_3$$
$$g_2 = (g_{2})_1 E_1 + (g_{2})_2 E_2 + (g_{2})_3 E_3$$
$$g_3 = (g_{3})_1 E_1 + (g_{3})_2 E_2 + (g_{3})_3 E_3$$

$$\tilde{F} = \begin{bmatrix}
(g_{1})_1 & (g_{2})_1 & (g_{3})_1 \\
(g_{1})_2 & (g_{2})_2 & (g_{3})_2 \\
(g_{1})_3 & (g_{2})_3 & (g_{3})_3 
\end{bmatrix}$$

$$F_{ij} = (g_j)_i$$

Fibre stretch:

$$\lambda = \frac{\|\tilde{m}\|}{\|\tilde{M}\|}$$
Deformation – Graphical Example I

Elongation into page, with no distortion.

\[
\begin{bmatrix}
(g_1)_1 & (g_2)_1 & (g_3)_1 \\
(g_1)_2 & (g_2)_2 & (g_3)_2 \\
(g_1)_3 & (g_2)_3 & (g_3)_3
\end{bmatrix}
\begin{bmatrix}
1.3 \\
-0.05 \\
0
\end{bmatrix}
\begin{bmatrix}
1.78 \\
1.27 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1.5
\end{bmatrix}
\]

\[
\hat{\epsilon}_2 = 1.78 \epsilon_1 + 1.27 \epsilon_2
\]

\[
\hat{\epsilon}_1 = 1.3 \epsilon_1 - 0.05 \epsilon_2
\]
Deformation – Graphical Example II

\[
[F] = \begin{bmatrix}
  \frac{1}{2} & 2 & 0 \\
  -1 & -\frac{1}{4} & 0 \\
  0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
  (g_1)_1 & (g_2)_1 & (g_3)_1 \\
  (g_1)_2 & (g_2)_2 & (g_3)_2 \\
  (g_1)_3 & (g_2)_3 & (g_3)_3 \\
\end{bmatrix}
\]

\[
g_1 = \frac{1}{2}E_1 - E_2
\]

\[
g_2 = 2E_1 - \frac{1}{4}E_2
\]

\[
g_3 = E_3
\]
Polar Decomposition of the Deformation Gradient Tensor

F = R • U

Rigid Body Rotation Tensor

Material vectors change orientation
BUT
They don’t change length

Proper orientations:
Determinant = +1

Stretch Tensor

There are 3 material directions (principal directions) in the initial configuration that:

- can change length
- BUT
do NOT change direction

Tensor is:
Positive definite
Symmetric
Determinant > 0

Material vectors change orientation BUT They don’t change length

There are 3 material directions (principal directions) in the initial configuration that:

- can change length
- BUT
do NOT change direction

Tensor is:
Positive definite
Symmetric
Determinant > 0
**Stretch Tensor**

\[ d\tilde{x} = \bar{F} \cdot d\tilde{X} = R \cdot U \cdot d\tilde{X} \]

\( U \cdot d\tilde{X} \) This expression quantifies the stretching of the material fibre \( d\tilde{X} \), as well as the part of the fibre rotation that results strictly from the material distortion (shape change).

The stretch tensor \( U \) possesses 3 real, positive eigenvalues (**principal values**). ➔ It is diagonal in its principal basis.

**The 3 eigenvectors are the principal directions.**

Material vectors along the principal directions may change length, but they do not change direction.

The 3 principal values equal the ratio of the deformed to the undeformed lengths of the three non-rotating material fibres.
Fibres aligned with the principal directions of $\mathbf{U}$ don’t rotate.

Fibres not aligned with the principal directions will change orientation, but for every fibre rotating one way, there will be another fibre rotating by the same amount in the opposite way.

The overall material rotation under a pure stretch is zero.

\[
\beta = \frac{1}{2} \left( \frac{\pi}{2} - \alpha \right)
\]
Strain

\[ \varepsilon = \frac{1}{2} (F^T \cdot F - I) \]

\[ [\varepsilon] = \frac{1}{2} \begin{pmatrix}
(g_1)_1 & (g_1)_2 & (g_1)_3 \\
(g_2)_1 & (g_2)_2 & (g_2)_3 \\
(g_3)_1 & (g_3)_2 & (g_3)_3
\end{pmatrix} \begin{pmatrix}
(g_1)_1 & (g_2)_1 & (g_3)_1 \\
(g_1)_2 & (g_2)_2 & (g_3)_2 \\
(g_1)_3 & (g_2)_3 & (g_3)_3
\end{pmatrix} - \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \]

\[ \varepsilon_{11} = \frac{1}{2} \left[ \left( (g_1)_1 \right)^2 + \left( (g_1)_2 \right)^2 + \left( (g_1)_3 \right)^2 \right] = \frac{1}{2} \left( \| g_1 \|^2 - 1 \right) \]

\[ = \frac{1}{2} \left[ (E_1 + \Delta E_1)^2 - 1 \right] = \frac{1}{2} \left[ (1)^2 + (\Delta E_1)^2 + 2\Delta E_1 - 1 \right] \approx \Delta E_1 \]

\[ \varepsilon_{11} \approx \frac{\Delta E_1}{E_1} = \frac{E_1 + \Delta E_1}{E_1} - \frac{E_1}{E_1} = \lambda - 1 \]

Fibre stretch

\[ \lambda = \frac{|m|}{\|M\|} \]

Normal strain
Strain

\[ \varepsilon_{12} = \frac{1}{2} g_1 g_2 \cos \alpha = \frac{1}{2} (E_1 + \Delta E_1)(E_2 + \Delta E_2) \cos \alpha = \frac{1}{2} (1 + \Delta E_1)(1 + \Delta E_2) \cos \alpha \]

\[ \approx \frac{1}{2} (1 + \text{small terms}) \cos \alpha \approx \frac{1}{2} \cos \alpha = \frac{1}{2} \sin \left( \frac{\pi}{2} - \alpha \right) \approx \frac{1}{2} \left( \frac{\pi}{2} - \alpha \right) \]

Shear strain (distortion)

\[ \beta = \frac{1}{2} \left( \frac{\pi}{2} - \alpha \right) \]
Elastic vs Plastic Strain

\[ \varepsilon_{\text{total}} = \varepsilon_{\text{elastic}} + \varepsilon_{\text{plastic}} \]

\[ \sigma = \varepsilon_{\text{elastic}} \cdot \text{slope} \]

\[ \sigma_2 > \sigma_1 \implies \varepsilon_{\text{elastic}(2)} > \varepsilon_{\text{elastic}(1)} \]
Elastic vs Plastic Strain

\[ \varepsilon_{\text{total}} = \varepsilon_{\text{elastic}} + \varepsilon_{\text{plastic}} \]

\[ \sigma_2 = \sigma_1 \implies \varepsilon_{\text{elastic}(2)} = \varepsilon_{\text{elastic}(1)} \]
Elastic Strain

Shape and/or size of the unit cell change.

Shape change is reversible.
Plastic Strain

Embedding blocks of crystal slide with respect to one another.

Shape and size of the unit cell DO NOT change.
Measuring Strain by Diffraction

\[ \varepsilon \equiv \frac{d - d_0}{d_0} = \frac{d}{d_0} - 1 \]

\[ \varepsilon = \frac{\sin \theta_0}{\sin \theta} - 1 \]

- We only measure normal strains – not shear strains!
- We measure lattice strains i.e. elastic strains
Mapping Strain - Direction and Gauge Volume

Measure *component of strain* parallel to bisector (Q)

Get strain data only from material inside the Gauge Volume
Reorient the sample to measure strain along different specimen directions.

Measure *component of strain* parallel to bisector (Q)

Get strain data only from material inside the Gauge Volume
The Gauge Volume

Fig. 5.18. Definition of the position and shape of the probe volume in a neutron experiment by the divergent and receiving slit/coller collimator combination (a), experiment geometry (b).

Fig. 5.19 a–c. Spatial resolution obtained by various divergent and receiving slit combinations. The dimensions of the probe region must be chosen such that there is negligible variation of residual stress within the probe volume.
Sampling a Strain Gradient

Large gauge volume smears out the gradient
Mapping Strain - Location, Location, Location!

Sampling volume

Incident beam

Diffracted beam

Spatial resolution
~ 1 mm³

Use neutrons to locate sample edges – Wall scans (0.1 mm)
**Definition of Stress**

**Stress** ($\sigma$) is a force per unit area.

The force can be uniformly distributed over an area:

$$\sigma = \frac{F}{A}$$

The force can be distributed non-uniformly over an area, in which case we define the stress at a point as:

$$\text{stress} = \lim_{dA \to 0} \frac{F}{dA}$$
Stress is a Tensor

Consider 2 planes passing through a point P in a cylindrical body in a state of simple tension.

The force acting on planes $A_1$ and $A_2$ is the same, but the stress on the two planes is different because the areas are different.

We need 2 vectors to define a stress:
- The force per unit area, $T$, acting on the plane
- The normal to the surface, $n$, on which $F$ acts

$\Rightarrow$ Stress is a 2nd order tensor that relates $T$ and $n$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$
Normal and Shear Stresses

The stress acting on a surface can be resolved into two components:
1) Normal stress - acts perpendicular to the surface
2) Shear stress - acts parallel to the surface

The direction of the normal stress on a plane is specified once the surface has been identified.

A shear stress can be further resolved into two components in the directions of the coordinate axes in the plane.

⇒ We need three components to define the state of stress at a point on a plane.
Components of Stress

We specify all the normal and shear stresses acting on the faces of an infinitesimal cube – a total of 18 components.

\[ \sigma_{ij} \Rightarrow \begin{align*} 
    i &= \text{coordinate parallel to which stress acts} \\
    j &= \text{coordinate plane normal to the plane acted on by the stress}
\end{align*} \]

\[ i = j \Rightarrow \text{normal stress.} \]
\[ i \neq j \Rightarrow \text{shear stress.} \]
Components of stress

We can show that $\sigma_{ij} = \sigma_{-i-j} \Rightarrow$ only 9 components

$$\sigma_{ij} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}$$

To satisfy equilibrium, we must have $\sigma_{ij} = \sigma_{ji} \Rightarrow 6$ independent components

$$\sigma_{ij} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix}$$

Symmetric tensor
Stress-Strain Relations

For small deformations in an isotropic material, there is a simple linear relationship between stress and strain.

\[ \varepsilon_{11} = \frac{1}{E} \left( \sigma_{11} - \nu (\sigma_{22} + \sigma_{33}) \right) \quad \varepsilon_{12} = \frac{\sigma_{12}}{2G} \]

This linear relationship arises because the force, \( F \), required for a displacement of \( u \) between 2 atoms, is given by the gradient of energy:

\[ F = \frac{dU}{dr} = \left( \frac{d^2U}{dr^2} \right)_{r_0} u \]

\[ \sigma = E \varepsilon \]
Getting Stress from Strain I

• The stress and strain tensors are related via the elastic stiffness tensor $\mathbf{C}$.

\[ \sigma_{ij} = \sum_{k} \sum_{l} C_{ijkl} \varepsilon_{kl} \]

\[ \mathbf{\sigma} = \mathbf{C} : \mathbf{\varepsilon} \]

• If the Principal Stress directions are known, we only need to measure the corresponding normal strain components.

• Then we can calculate the 3 Principal Stresses from the 3 strains using the Generalized Hooke’s Law (assuming elastic isotropy):

\[ \sigma_\alpha = \frac{E}{1+\nu} \left[ \varepsilon_\alpha + \frac{\nu}{1-2\nu} (\varepsilon_x + \varepsilon_y + \varepsilon_z) \right], \alpha = (x, y, z) \]
Getting Stress from Strain II

If the *Principal Stress Directions* are *not* known, we need to measure enough strain components to determine the full *strain tensor* $\Rightarrow$ *stress tensor*

\[
\begin{align*}
\varepsilon_a &= a_m a_n \varepsilon_{mn} \quad a_m = a \cdot E_m \\
\varepsilon_b &= b_m b_n \varepsilon_{mn} \quad b_m = b \cdot E_m \\
\end{align*}
\]

*6 equations in 6 unknowns*

\[
\begin{align*}
\varepsilon_f &= f_m f_n \varepsilon_{mn} \quad f_m = f \cdot E_m \\
\end{align*}
\]

\[
\varepsilon_a = \frac{\sin \theta_0}{\sin \theta} - 1
\]

\[
\sigma_{ij} = \sum_{k} \sum_{l} C_{ijkl} \varepsilon_{kl}
\]
Residual Stresses – Good or Bad

• Residual stresses are self-equilibrating stresses within a stationary solid body when no external forces are applied.

• They vary with location in a component: consider a cast plate a few mm thick. The stresses in the core balance those in the skin. What happens if the skin is removed on one of the surfaces?

  Inner core is in tension  \[\rightarrow\]  Outer skin is in compression

• Almost every manufactured component has residual stresses – they can be detrimental or beneficial to the performance of the component.

• They are not apparent, and they are difficult to measure and predict.

• We need a reliable method to measure residual stresses non-destructively.
Residual Stresses Can Kill You

Liquid-Metal-Assisted Cracking of a plain carbon steel I-beam after hot-dip galvanizing. The crack is 1.1 m long, and appeared after the beam had been in service.
Internal stresses are categorized according to the length scale over which they vary.

Type I stresses - macrostresses - vary slowly over the volume of a component and are considered to be continuous across grain boundaries and phase boundaries.

These are the stresses calculated during a typical stress analysis of an engineering component (e.g. finite element analysis).
Type II microstresses vary on a length scale on the order of the grain size.

Type III microstresses vary on a subgrain length scale.

Neutron diffraction can be used to study macrostresses and type II microstresses.

Type III microstresses typically give rise to peak broadening.
Evaluating Processes
Heat-Exchanger Twist Tube

- Customer’s question: “Does our annealing procedure reduce the residual stresses?”
- Largest stresses due to deformation expected to be in the hoop direction
- Need only measure hoop strain on As-Manufactured and Annealed tubes

Circumferential scan of 24 positions
3 through-thickness positions:
- mid-thickness
- 0.4 mm toward OD
- 0.4 mm toward ID
Heat-Exchanger Twist Tube
As-Manufactured

• Significant strain variations are observed.
• Complementary OD and ID variations.
Heat-Exchanger Twist Tube
Annealed

Annealed Twist Tube

Clearly, annealing effectively reduced the residual stresses
In-Situ Deformation Experiments

- Measuring residual strains and deformation textures is great – but how did they get there?

- In-situ experiments allow us to follow the evolution of residual strains and texture during deformation.

- Example: Deformation of a Mg-Al alloy
Plastic Deformation: Slip vs. Twinning

- Slip: undeformed blocks of crystal translate relative to each other
- Twinning: homogeneous shear of a portion of the lattice (mirror)
Plastic Deformation: Slip

(0001)<\{11\overline{2}0}\rangle \rightarrow 2
CRSS = 0.51 \text{ MPa}

\{10\overline{1}0\}<\{11\overline{2}0\rangle \rightarrow 2
CRSS = 40 \text{ MPa}

\{10\overline{1}1\}<\{11\overline{2}0\rangle \rightarrow 4
CRSS = 40 \text{ MPa} ??
Plastic Deformation: Twinning

- Polar: only limited $c$-axis extension (6.9% max)
• Slip: undeformed blocks of crystalline material slide past one another
  – Gradual lattice reorientation
• Twinning: Abrupt reorientation of the crystal lattice
Mg - 8.5 wt.% Al

- Processing (Pechiney, France)
  - Mix Mg and Al under protective atmosphere
  - Cast into billets
  - Extrude at 250°C (64 mm → 15 mm)
  - Age

Centre is extrusion axis
Contour separation = 0.5 × random
Texture Modification

Thermomechanical Treatment

Compress // extrusion axis

\{1012\} twinning leads to an 86.3° reorientation of the lattice

Anneal
Texture Modification

Contour separation: 1 × uniform
Extrusion texture (Texture 1)
Modified texture (Texture 2)

Contour separation: 2.5 × uniform
Cyclic Tension

- Hysteresis loops occur for both textures
Metallography

Texture 1
Initial State

Texture 2
Texture 1

Texture 2

Cycle 6
Texture 1:
- Twins are narrow and there are relatively few of them

Texture 2:
- Long, wide twins which cross grain boundaries
- Much smaller twins, apparently nucleated at grain boundaries

Texture 2:
- Twins are narrow and there are relatively few of them
Cyclic Tension - Texture 2

- Profuse \{1012\} tension twinning expected in a majority of grains
Cyclic Tension - Texture 2

Stress (MPa)

{1012} axial

Lattice strain (µstrain)

Integrated intensity

Stress (MPa)
Cyclic Tension - Texture 2

\[ \sigma \] 47°
Cyclic Tension - Texture 1

Stress (MPa) vs. Lattice strain (µstrain)

Stress (MPa) vs. Integrated intensity

(0002) axial
Cyclic Tension - Texture 1

(0002) axial

Stress (MPa) vs. Lattice strain (µstrain)

Stress (MPa) vs. Integrated intensity
Cyclic Tension - Texture 1

Stress (MPa)

Lattice strain (μstrain)

Integrated intensity

(0002) axial

σ

σ

0°
Twinning - Stress Relaxation

Single crystal
Twinning - Stress Relaxation
Polycrystal
Twinning - Stress Relaxation
Polycrystal

Local stress relaxation $\Rightarrow$ grain level